

NON-LOCAL PDEs WITH DISCRETE STATE-DEPENDENT DELAYS: WELL-POSEDNESS IN A METRIC SPACE ¹

ALEXANDER V. REZOUNENKO

Department of Mechanics and Mathematics

V.N.Karazin Kharkiv National University, 4, Svobody Sqr., Kharkiv, 61077, Ukraine

Email: rezounenko@univer.kharkov.ua

PETR ZAGALAK

Institute of Information Theory and Automation

Academy of Sciences of the Czech Republic, P.O. Box 18, 182 08 Praha, Czech Republic

Email: zagalak@utia.cas.cz

Abstract. Partial differential equations with discrete (concentrated) state-dependent delays are studied. The existence and uniqueness of solutions with initial data from a wider linear space is proven first and then a subset of the space of continuously differentiable (with respect to an appropriate norm) functions is used to construct a dynamical system. This subset is an analogue of *the solution manifold* proposed for ordinary equations in [H.-O. Walther, The solution manifold and C^1 -smoothness for differential equations with state-dependent delay, J. Differential Equations, 195(1), (2003) 46–65]. The existence of a compact global attractor is proven.

1 Introduction

The partial differential equations (PDEs) with delays have attracted a lot of attention during the last decades as many processes of the real world (like an automatically controlled furnace, bi-directional associative memory (BAM) neural networks, reaction-diffusion processes) can be described by such kind of equations. Studying these equations is based on the well-developed approaches to the ordinary differential equations (ODEs) with delays [11, 7, 1] and PDEs without delays [8, 9, 15, 14]. Under certain assumptions both types of equations describe a kind of dynamical systems that are infinite-dimensional, see [2, 30, 6] and references therein; see also [31, 4, 5, 3] and to the monograph [37] that are very close to this work.

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In many evolution systems arising in applications the presented delays are frequently *state-dependent* (SDDs). The theory of such equations, especially the ODEs, is rapidly developing and many deep results have been obtained up to now (see e.g. [32, 33, 34, 16, 18, 35] and also the survey paper [12] for details and references). The underlying main mathematical difficulty of the theory of PDEs with SDDs lies in the fact that the functions describing state-dependent delays are not Lipschitz continuous on the space of continuous functions - the main space, on which the classical theory of equations with delays is developed. This implies that the corresponding *initial value problem* (IVP) is not generally well-posed in the sense of J. Hadamard [8, 9].

The partial differential equations with state-dependent delays were first studied in [21] (the case of distributed delays, weak solutions), [13] (mild solutions, infinite discrete delay), and [22] (weak solutions, finite discrete and distributed delays). An alternative approach to the PDEs with discrete SDDs is proposed in [24].

This paper is a continuation of the work [25] and its goal is to study the approach used for ODEs with SDDs [32, 33, 12] in the case of PDEs. The main idea lies in finding a wider space $Y \supset X$ such that a solution $u : [a, b] \rightarrow Y$ be a Lipschitz function (with respect to a weaker norm of Y), and constructing a dynamical system on a subset of the space $C([a, b]; Y)$. It should be emphasized that the dynamical system is constructed on a metric space that is nonlinear. More precisely, the existence and uniqueness of solutions with initial data from a wider linear space is proven first and then a subset of the space of continuously differentiable (with respect to an appropriate norm) functions is used to construct the aforementioned dynamical system. This subset is an analogue of *the solution manifold* proposed in [33], see also [12]. We use the same class of non-local in space variables nonlinear PDEs as in [25].

The paper is organized as follows. The section 2 is devoted to the formulation of the model. The proof of the existence and uniqueness of (strong) solutions for initial functions from a Banach space forms a main part of the section 3. In the section 4, an evolution operator S_t is constructed and its asymptotic properties in different functional spaces are investigated. The dissipativeness is obtained in a Banach space, while the existence of a global attractor is proven on a smaller metric space (the solution manifold). The choice of this smaller space is different from that proposed in [25].

2 The model with discrete state-dependent delay and preliminaries

Consider the following non-local partial differential equation with a discrete state-dependent delay η

$$\frac{\partial}{\partial t}u(t, x) + Au(t, x) + du(t, x) = b([Bu(t - \eta(u_t), \cdot)](x)) \equiv (F_1(u_t))(x), \quad x \in \Omega, \quad (1)$$

where A is a densely-defined self-adjoint positive linear operator with domain $D(A) \subset L^2(\Omega)$ and compact resolvent, which means that $A : D(A) \rightarrow L^2(\Omega)$ generates an analytic semigroup, $\Omega \subset \mathbb{R}^{n_0}$ is a smooth bounded domain, $B : L^2(\Omega) \rightarrow L^2(\Omega)$ denotes a bounded operator that will be defined later, $b : \mathbb{R} \rightarrow \mathbb{R}$ stands for a locally Lipschitz map, $d \in \mathbb{R}$, $d \geq 0$, and the function $\eta : C([-r, 0]; L^2(\Omega)) \rightarrow [0, r] \subset \mathbb{R}_+$ denotes a *state-dependent discrete delay*. Let $C \equiv C([-r, 0]; L^2(\Omega))$. Norms defined on $L^2(\Omega)$ and C are denoted by $\|\cdot\|$ and $\|\cdot\|_C$, respectively, and $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(\Omega)$. As usually, $u_t \equiv u_t(\theta) \equiv u(t + \theta)$ for $\theta \in [-r, 0]$.

Remark 1. The operator B may for example be of the following forms (linear operators)

$$[Bv](x) \equiv \int_{\Omega} v(y) \tilde{f}(x, y) dy, \quad x \in \Omega, \quad (2)$$

or even simpler

$$[Bv](x) \equiv \int_{\Omega} v(y) f(x - y) \ell(y) dy, \quad x \in \Omega, \quad (3)$$

where $f : \Omega \rightarrow \mathbb{R}$ is a smooth function and $\ell \in C_0^\infty(\Omega)$. In the last case the nonlinear term in (1) is of the form

$$(F_1(u_t))(x) \equiv b\left(\int_{\Omega} u(t - \eta(u_t), y) f(x - y) \ell(y) dy\right), \quad x \in \Omega. \quad (4)$$

□

Consider the equation (1) with the initial condition

$$u|_{[-r, 0]} = \varphi \quad (5)$$

and let

$$H \equiv \left\{ \varphi \in C([-r, 0]; D(A^{-\frac{1}{2}})) \mid \varphi(0) \in D(A^{\frac{1}{2}}) \right\}. \quad (6)$$

Let further

$$\|\varphi\|_H \equiv \max_{s \in [-r, 0]} \|A^{-\frac{1}{2}}\varphi(s)\| + \|A^{\frac{1}{2}}\varphi(0)\|$$

be a norm defined on the space H and $D(A^\alpha)$ denote the domain of the operator A^α . In the sequel the following assumptions will play an important role.

(H1. η) The discrete delay function $\eta : H \rightarrow [0, r]$ is such that

$$\begin{aligned} \exists L_\eta > 0, \quad \exists q \geq 0 \quad \text{such that} \quad \forall \varphi, \psi \in H \Rightarrow \\ |\eta(\varphi) - \eta(\psi)| \leq L_\eta \left(q \|A^{\frac{1}{2}}(\varphi(0) - \psi(0))\|^2 + \int_{-r}^0 \|A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))\|^2 d\theta \right)^{\frac{1}{2}} \end{aligned} \quad (7)$$

(H.B) The following Lipschitz property of the operator B holds.

$$\exists L_B > 0 \text{ such that } \forall u, v \in D(A^{-\frac{1}{2}}) \Rightarrow \|Bu - Bv\| \leq L_B \|A^{-\frac{1}{2}}(u - v)\| \quad (8)$$

Remark 2. Under the assumption that for all (almost all) $x \in \Omega \Rightarrow f(\cdot - x)\ell(\cdot) \in D(A^{\frac{1}{2}})$ and $u \in D(A^{-\frac{1}{2}})$, the term of the form (3) implies that

$$|\langle u, f(\cdot - x)\ell(\cdot) \rangle| \leq \|A^{-\frac{1}{2}}u\| \|A^{\frac{1}{2}}f(\cdot - x)\ell(\cdot)\|,$$

which gives

$$\left(\int_{\Omega} \left| \int_{\Omega} u(y) f(y - x) \ell(y) dy \right|^2 dx \right)^{\frac{1}{2}} \leq \|A^{-\frac{1}{2}}u\| \left(\int_{\Omega} \|A^{\frac{1}{2}}f(\cdot - x)\ell(\cdot)\|^2 dx \right)^{\frac{1}{2}}.$$

Hence, the property (H.B) (see (8)) holds with $L_B \equiv \left(\int_{\Omega} \|A^{\frac{1}{2}}f(\cdot - x)\ell(\cdot)\|^2 dx \right)^{\frac{1}{2}}$. The same arguments hold (with $L_B \equiv \left(\int_{\Omega} \|A^{\frac{1}{2}}\tilde{f}(x, \cdot)\|^2 dx \right)^{\frac{1}{2}}$) for a more general term of the form (2). \square

Let now the following space

$$\mathcal{L} \equiv \left\{ \varphi \in C([-r, 0]; D(A^{-\frac{1}{2}})) \mid \sup_{s \neq t} \left\{ \frac{\|A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))\|}{|s - t|} \right\} < +\infty; \varphi(0) \in D(A^{\frac{1}{2}}) \right\}, \quad (9)$$

with the natural norm

$$\|\varphi\|_{\mathcal{L}} \equiv \max_{s \in [-r, 0]} \|A^{-\frac{1}{2}}\varphi(s)\| + \sup_{s \neq t} \left\{ \frac{\|A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))\|}{|s - t|} \right\} + \|A^{\frac{1}{2}}\varphi(0)\| \quad (10)$$

be defined. For any segment $[a, b] \subset \mathbb{R}$ (c.f. (9)) and any Lipschitz-on- $[a, b]$ function φ , let

$$|||\varphi|||_{[a,b]} \equiv \sup \left\{ \frac{\|A^{-\frac{1}{2}}(\varphi(s) - \varphi(t))\|}{|s - t|} : s \neq t; s, t \in [a, b] \right\} \quad (11)$$

denote its Lipschitz constant and let $|||\varphi||| \equiv |||\varphi|||_{[-r,0]}$. Then the following lemma holds.

Lemma 1. Let the assumptions (H1. η) and (H.B) hold (see (7), (8)) and let the function $b : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz and bounded ($|b(s)| \leq M_b$ for all $s \in \mathbb{R}$). Then any two functions $\varphi \in \mathcal{L}, \psi \in H$ (with H and \mathcal{L} defined in (6) and (9)) the nonlinearity F satisfies

$$\|F_1(\varphi) - F_1(\psi)\| \leq L_{F_1} \left[|||\varphi||| \right] (q \|A^{1/2}(\varphi(0) - \psi(0))\| + \|A^{-1/2}(\varphi - \psi)\|_C), \quad (12)$$

where

$$L_{F_1}[\ell] \equiv L_b L_B \sqrt{2} \max \{1; \ell L_\eta \max\{1; \sqrt{r}\}\} \quad (13)$$

and $L_{F_1}[\ell]$ is used in (12) with

$$\ell = L_\varphi \equiv |||\varphi||| \equiv \sup \left\{ \frac{\|A^{-1/2}(\varphi(s) - \varphi(t))\|}{|s - t|} : s \neq t; s, t \in [-r, 0] \right\}.$$

Proof of Lemma 1. Using the Lipschitz property of b and B (see (H.B)), it follows that

$$\begin{aligned} \|F_1(\varphi) - F_1(\psi)\|^2 &= \int_{\Omega} |b([B\varphi](-\eta(\varphi), x)) - b([B\psi](-\eta(\psi), x))|^2 dx \leq \\ &\leq L_b^2 \int_{\Omega} |[B\varphi](-\eta(\varphi), x) - [B\psi](-\eta(\psi), x)|^2 dx = L_b^2 \|[B\varphi](-\eta(\varphi), \cdot) - [B\psi](-\eta(\psi), \cdot)\|^2 \leq \\ &\leq L_b^2 L_B^2 \|A^{-1/2} \{\varphi(-\eta(\varphi)) - \psi(-\eta(\psi)) \pm \varphi(-\eta(\psi))\}\|^2 \leq \\ &\leq 2L_b^2 L_B^2 (\|A^{-1/2} \{\varphi(-\eta(\varphi)) - \varphi(-\eta(\psi))\}\|^2 + \|A^{-1/2}(\varphi - \psi)\|_C^2). \end{aligned}$$

Next, $\varphi \in \mathcal{L}$ implies that there exists $L_\varphi \equiv |||\varphi||| > 0$, (see (10),(11)) such that

$$\|A^{-1/2}(\varphi(s^1) - \varphi(s^2))\| \leq L_\varphi |s^1 - s^2|, \quad \forall s^1, s^2 \in [-r, 0]. \quad (14)$$

Hence, (14) and (H1. η) give

$$\|F_1(\varphi) - F_1(\psi)\|^2 \leq$$

$$\begin{aligned}
&\leq 2L_b^2 L_B^2 \left[L_\varphi^2 L_\eta^2 \left(q \|A^{1/2}(\varphi(0) - \psi(0))\|^2 + \int_{-r}^0 \|A^{-\frac{1}{2}}(\varphi(\theta) - \psi(\theta))\|^2 d\theta \right) + \right. \\
&\quad \left. + \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right] \leq \\
&\leq 2L_b^2 L_B^2 \left[L_\varphi^2 L_\eta^2 \left(q \|A^{1/2}(\varphi(0) - \psi(0))\|^2 + r \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right) + \right. \\
&\quad \left. + \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right] \leq \\
&\leq 2L_b^2 L_B^2 \max\{1; L_\varphi^2 L_\eta^2 \max\{1; r\}\} \left[q \|A^{1/2}(\varphi(0) - \psi(0))\|^2 + \|A^{-\frac{1}{2}}(\varphi - \psi)\|_C^2 \right].
\end{aligned}$$

The last estimate and using the formulas $\sqrt{\max\{|a|; |b|\}} = \max\{\sqrt{|a|}; \sqrt{|b|}\}$ and $\sqrt{a^2 + b^2} \leq |a| + |b|$ give (12), (13), which completes the proof. \square

3 The existence and uniqueness of solutions

As in [25] we need the following

Definition 1. A vector-function $u(t) \in C([-r, T]; D(A^{-1/2})) \cap C([0, T]; D(A^{1/2})) \cap L^2(0, T; D(A))$ with derivative $\dot{u}(t) \in L^\infty(0, T; D(A^{-1/2}))$ is a (strong) solution to the problem defined by (1) and (5) on $[0, T]$ if

(a) $u(\theta) = \varphi(\theta)$ for $\theta \in [-r, 0]$;

(b) $\forall v \in L^2(0, T; L^2(\Omega))$ such that $\dot{v} \in L^2(0, T; D(A^{-1}))$ and $v(T) = 0 \Rightarrow$

$$\begin{aligned}
& - \int_0^T \langle u(t), \dot{v}(t) \rangle dt + \int_0^T \langle A^{1/2}u(t), A^{1/2}v(t) \rangle dt = \\
& = \langle \varphi(0), v(0) \rangle + \int_0^T \langle F_1(u_t) - d \cdot u(t), v(t) \rangle dt. \quad (15)
\end{aligned}$$

Now we prove the following theorem on the existence and uniqueness of solutions.

Theorem 1. Let the assumptions (H1. η) and (H.B) hold and let the function $b : \mathbb{R} \rightarrow \mathbb{R}$ be Lipschitz and bounded, i.e. $|b(s)| \leq M_b$ for all $s \in \mathbb{R}$. Let further $\varphi \in \mathcal{L}$ be a given initial condition. Then the problem defined by (1) and (5) has a unique solution on any time interval $[0, T]$ such that $\dot{u} \in L^2(0, T; L^2(\Omega))$.

Remark 3. Notice that φ does not assume $\varphi \in L^2([-r, 0]; D(A))$. However, the definition of a strong solution implies that

$$u_t \in L^2([-r, 0]; D(A)), \quad \forall t \geq r. \quad (16)$$

□

Proof of Theorem 1. We follow the proof of Theorem 1 in [25]. Notice that the assumption (H1. η) is slightly more general than the assumption (H. η) in [25]. This implies some changes in the proof of the uniqueness of solutions.

Let $\{e_k\}_{k=1}^\infty$ denote an orthonormal basis of $L^2(\Omega)$ such that $Ae_k = \lambda_k e_k$, $0 < \lambda_1 < \dots < \lambda_k \rightarrow +\infty$ and consider the Galerkin approximate solution $u^m = u^m(t, x) = \sum_{k=1}^m g_{k,m}(t) e_k$ of order m such that

$$\begin{cases} \langle \dot{u}^m + Au^m + du^m - F_1(u_t^m), e_k \rangle = 0, \\ \langle u^m(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle, \quad \forall \theta \in [-r, 0] \end{cases} \quad (17)$$

$\forall k = 1, \dots, m$, $g_{k,m} \in C^1(0, T; \mathbb{R}) \cap L^2(-r, T; \mathbb{R})$ with $\dot{g}_{k,m}(t)$ absolutely continuous.

The system (17) is a system of (ordinary) differential equations in \mathbb{R}^m with a concentrated (discrete) state-dependent delay for the unknown vector function $U(t) \equiv (g_{1,m}(t), \dots, g_{m,m}(t))$ (for the corresponding theory see [33, 34] and also a recent review [12]).

The key difference between equations with state-dependent and state-independent (concentrated) delays is that the first type of equations is not well-posed in the space of continuous (initial) functions. To get a well-posed initial value problem, it is better [33, 34, 12] to use a smaller space of Lipschitz continuous functions or even a smaller subspace of $C^1([-r, 0]; \mathbb{R}^m)$.

The condition $\varphi \in \mathcal{L}$ implies that the function $U(\cdot)|_{[-r, 0]} \equiv P_m \varphi(\cdot)$, which defines initial data, is Lipschitz continuous as a function from $[-r, 0]$ to \mathbb{R}^m . Here P_m is the orthogonal projection onto the subspace $\text{span}\{e_1, \dots, e_m\} \subset L^2(\Omega)$. Hence, we can apply the theory of ODEs with discrete state-dependent delay (see e.g. [12]) to get the local existence and uniqueness of solutions to (17).

Next, we will get an a priory estimate to prove the continuation of solutions u^m to (17) on any time interval $[0, T]$ and then use it for the proof (by the method of compactness, see [15]) of the existence of strong solutions to (1) and (5). To that end, multiply the first

equation in (17) by $\lambda_k g_{k,m}$ and sum for $k = 1, \dots, m$ to get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{1/2} u^m(t)\|^2 + \|Au^m(t)\|^2 + d \cdot \|A^{1/2} u^m(t)\|^2 &= \langle P_m F(u_t^m), Au^m(t) \rangle \leq \\ &\leq \frac{1}{2} \|P_m F(u_t^m)\|^2 + \frac{1}{2} \|Au^m(t)\|^2. \end{aligned}$$

As the function b is bounded, $\|F(u_t^m)\|^2 \leq M_b^2 |\Omega|$ (here $|\Omega| \equiv \int_{\Omega} 1 \, dx$), which gives

$$\frac{d}{dt} \|A^{1/2} u^m(t)\|^2 + \|Au^m(t)\|^2 \leq M_b^2 |\Omega|. \quad (18)$$

Integrating (18) with respect to t and using the relationships $\varphi(0) \in D(A^{1/2})$, $u^m(0) = P_m \varphi(0) \in D(A^{1/2})$, $\|A^{1/2} u^m(0)\| = \|A^{1/2} P_m \varphi(0)\| \leq \|A^{1/2} \varphi(0)\|$, we get an a priori estimate

$$\|A^{1/2} u^m(t)\|^2 + \int_0^t \|Au^m(\tau)\|^2 \, d\tau \leq \|A^{1/2} \varphi(0)\|^2 + M_b^2 |\Omega| T, \quad \forall m, \forall t \in [0, T]. \quad (19)$$

The above relationship (19) means that

$$\{u^m\}_{m=1}^{\infty} \text{ is a bounded set in } L^{\infty}(0, T; D(A^{1/2})) \cap L^2(0, T; D(A)).$$

Using this fact and (17), it follows that

$$\{\dot{u}^m\}_{m=1}^{\infty} \text{ is a bounded set in } L^{\infty}(0, T; D(A^{-1/2})) \cap L^2(0, T; L^2(\Omega)).$$

Hence, the family $\{(u^m; \dot{u}^m)\}_{m=1}^{\infty}$ is a bounded set in

$$\begin{aligned} Z_1 \equiv & (L^{\infty}(0, T; D(A^{1/2})) \cap L^2(0, T; D(A))) \times \\ & \times (L^{\infty}(0, T; D(A^{-1/2})) \cap L^2(0, T; L^2(\Omega))). \end{aligned} \quad (20)$$

Therefore, there exist a subsequence $\{(u^k; \dot{u}^k)\}$ and an element $(u; \dot{u}) \in Z_1$ such that

$$\{(u^k; \dot{u}^k)\} \text{ *-weak converges to } (u; \dot{u}) \text{ in } Z_1. \quad (21)$$

The proof that any *-weak limit is a strong solution is standard. To prove the property $u(t) \in C([0, T]; D(A^{1/2}))$, we use the well-known (see also [14, thm. 1.3.1])

Proposition 1. (Proposition 1.2 in [26]). Let V denote a dense Banach space that is continuously embedded in a Hilbert space X and let $X = X^*$ so that $V \hookrightarrow X \hookrightarrow V^*$. Then the Banach space $W_p(0, T) \equiv \{u \in L^p(0, T; V) : \dot{u} \in L^q(0, T; V^*)\}$ (here $p^{-1} + q^{-1} = 1$) is contained in $C([0, T]; X)$.

In our case $X = D(A^{1/2})$, $V = D(A)$, $V^* = L^2(\Omega)$, $p = q = 1/2$.

Now we prove the uniqueness of solutions. Using the fact that $\varphi \in \mathcal{L}$, the definition 1 of a solution v , and $\dot{v}(t) \in L^\infty(0, T; D(A^{-1/2}))$ (see (21)), it follows that for any such a solution v and any $T > 0$, there exists $L_{v,T} > 0$, such that

$$\|A^{-1/2}(v(s^1) - v(s^2))\| \leq L_{v,T}|s^1 - s^2|, \quad \forall s^1, s^2 \in [-r, T]. \quad (22)$$

In the light of (11), let $L_{v,T} \equiv \|v\|_{[-r, T]}$.

Consider any two solutions u and v of (1), (5) (not necessarily with the same initial function). The standard variation-of-constants formula $u(t) = e^{-At}u(0) + \int_0^t e^{-A(t-\tau)}F(u_\tau) d\tau$ and the estimate $\|A^\alpha e^{-tA}\| \leq \left(\frac{\alpha}{t}\right)^\alpha e^{-\alpha}$ (see e.g. [6, (1.17), p.84]) give

$$\begin{aligned} \|A^{1/2}(u(t) - v(t))\| &\leq e^{-\lambda_1 t} \|A^{1/2}(u(0) - v(0))\| + \int_0^t \|A^{1/2}e^{-A(t-\tau)}\| \|F(u_\tau) - F(v_\tau)\| d\tau \leq \\ &\leq e^{-\lambda_1 t} \|A^{1/2}(u(0) - v(0))\| + \int_0^t \left(\frac{1/2}{t-\tau}\right)^{1/2} e^{-1/2} \|F(u_\tau) - F(v_\tau)\| d\tau, \end{aligned} \quad (23)$$

as $\|A^{1/2}e^{-A(t-\tau)}\| \leq \left(\frac{1/2}{t-\tau}\right)^{1/2} e^{-1/2}$ and similarly,

$$\|A^{-1/2}(u_t - v_t)\|_C \leq \|A^{-1/2}(u_0 - v_0)\|_C + \int_0^t \|F(u_\tau) - F(v_\tau)\| d\tau.$$

The last estimate and (23) give (just the case when $q = 1$ is shown for the purpose of clarity)

$$\begin{aligned} \|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C &\leq e^{-\lambda_1 t} \|A^{1/2}(u(0) - v(0))\| + \\ &+ \|A^{-1/2}(u_0 - v_0)\|_C + \int_0^t \{1 + (2e(t-\tau))^{-1/2}\} \|F(u_\tau) - F(v_\tau)\| d\tau. \end{aligned} \quad (24)$$

It follows, from Lemma 1, that

$$\|F(u_t) - F(v_t)\| \leq L_{F_1, v, T} (q \|A^{1/2}(u(t) - v(t))\| + \|A^{-1/2}(u_t - v_t)\|_C), \quad (25)$$

where $L_{F_1, v, T}$ is defined in the same way as L_{F_1} in (13), just with $\ell = L_{v, T}$ instead of L_φ - see (13) and (22).

$$L_{F_1, v, T} \equiv L_{F_1} \left[L_{v, T} \right] \equiv L_b L_B \sqrt{2} \max \{1; L_{v, T} L_\eta \max \{1; \sqrt{r}\}\}. \quad (26)$$

It should be emphasized how the Lipschitz constant $L_{v,T} \equiv |||v|||_{[-r,T]}$ of a strong solution v is taken into account in (26) (see (22) and (11)).

Let

$$g(t) \equiv ||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C.$$

Then the relationships (24) and (25) lead to the following estimate

$$g(t) \leq g(0) + \int_0^t \{1 + (2e(t - \tau))^{-1/2}\} L_{F_1,v,T} \cdot g(\tau) d\tau$$

Lemma 2 (Gronwall). Let $u, \alpha \in C[a, b]$, $\beta(t) \geq 0$, β is integrable on $[a, b]$ and

$$u(t) \leq \alpha(t) + \int_a^t \beta(\tau) u(\tau) d\tau, \quad a \leq t \leq b$$

Then

$$u(t) \leq \alpha(t) + \int_a^t \beta(\tau) \alpha(\tau) \exp \left\{ \int_\tau^t \beta(s) ds \right\} d\tau, \quad a \leq t \leq b$$

Moreover, if α is non-decreasing, then

$$u(t) \leq \alpha(t) \exp \left\{ \int_a^t \beta(s) ds \right\}, \quad a \leq t \leq b.$$

It follows, from the above lemma and equality $\int_0^t (t - \tau)^{-1/2} d\tau = 2t^{1/2}$, that

$$\begin{aligned} g(t) &\leq g(0) \exp \left\{ L_{F_1,v,T} \int_0^t \{1 + (2e(t - s))^{-1/2}\} ds \right\} \leq \\ &\leq g(0) \exp \left\{ L_{F_1,v,T} \left(t + \sqrt{\frac{2t}{e}} \right) \right\}, \end{aligned}$$

which implies, $\forall t \in [0, T]$, that

$$\begin{aligned} &||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C \leq \\ &\leq E_{F_1,v,T} (||A^{1/2}(u(0) - v(0))|| + ||A^{-1/2}(u_0 - v_0)||_C), \end{aligned} \quad (27)$$

where

$$E_{F_1,v,T} \equiv \exp \left\{ L_{F_1,v,T} \cdot \left(T + \sqrt{\frac{2T}{e}} \right) \right\}, \quad (28)$$

see (26) for the definition of $L_{F_1,v,T} \equiv L_{F_1} [L_{v,T}]$. This proves the uniqueness of the solution to (1) and (5), and completes the proof of theorem 1. \square

4 Asymptotic properties of solutions

This section is devoted to studies of the asymptotic behavior of solutions in different functional spaces. We define first (in a standard way) the evolution semigroup $S_t : \mathcal{L} \rightarrow \mathcal{L}$ (the space \mathcal{L} is defined in (9)) by the formula

$$S_t \varphi \equiv u_t, \quad t \geq 0, \quad (29)$$

where $u(t)$ is a unique solution to the problem (1) and (5) (see definition 1).

The estimate (27) means the continuity of the evolution operator S_t in the norm of the space H (see (6)), i.e.

$$\|S_t \varphi - S_t \psi\|_H \leq E_{F_1, v, T} \|\varphi - \psi\|_H \text{ for all } t \in [0, T]. \quad (30)$$

The aim now is to get a more precise estimate, e.g. the continuity of S_t in the norm of the space \mathcal{L} (see (9), (10)). Consider the definition of the Galerkin approximate solution (see (17)). It gives

$$\begin{aligned} \|A^{-1/2}(\dot{u}^m(t) - \dot{v}^m(t))\| &\leq \|A^{1/2}(u^m(t) - v^m(t))\| + d\|A^{-1/2}(u^m(t) - \\ &\quad - v^m(t))\| + \|F_1(u_t^m) - F_1(v_t^m)\| \end{aligned}$$

and Lemma 1 implies

$$\begin{aligned} \|A^{-1/2}(\dot{u}^m(t) - \dot{v}^m(t))\| &\leq (1 + d + L_{F_1})\{\|A^{1/2}(u^m(t) - v^m(t))\| + \\ &\quad + \|A^{-1/2}(u_t^m - v_t^m)\|_C\}. \end{aligned}$$

An analogous estimate for a solution to the problem (1) and (5), can be obtained from (21) and the following

Proposition 2. [38, Theorem 9] Let X be a Banach space. Then any *-weak convergent sequence $\{w_k\}_{n=1}^\infty \in X^*$ *-weak converges to an element $w_\infty \in X^*$ and $\|w_\infty\|_X \leq \liminf_{n \rightarrow \infty} \|w_n\|_X$.

More precisely,

$$\begin{aligned} \text{ess sup}_{t \in [0, T]} \|A^{-1/2}(\dot{u}(t) - \dot{v}(t))\| &\leq (1 + d + L_{F_1}) \sup_{t \in [0, T]} \{\|A^{1/2}(u(t) - v(t))\| + \\ &\quad + \|A^{-1/2}(u_t - v_t)\|_C\} \end{aligned}$$

The last estimate and relationship (27) imply

$$\begin{aligned} \operatorname{ess\,sup}_{t \in [0, T]} \|A^{-1/2}(\dot{u}(t) - \dot{v}(t))\| &\leq \\ &\leq (1 + d + L_{F1})E_{F1, v, T} (\|A^{1/2}(u(0) - v(0))\| + \|A^{-1/2}(u_0 - v_0)\|_C) \end{aligned} \quad (31)$$

Hence, see (11),

$$\|u - v\|_{[0, T]} \leq (1 + d + L_{F1})E_{F1, v, T} (\|A^{1/2}(u(0) - v(0))\| + \|A^{-1/2}(u_0 - v_0)\|_C)$$

From that and (27), it follows that

$$\|u_t - v_t\|_{\mathcal{L}} \leq (2 + d + L_{F1})E_{F1, v, T} \|u_0 - v_0\|_{\mathcal{L}}, \quad \forall t \in [0, T], \quad (32)$$

which finally means that for any $T \geq 0$ there exists a constant $C_T > 0$ such that $\forall t \in [0, T]$ it gives

$$\|u_t - v_t\|_{\mathcal{L}} = \|S_t \varphi - S_t \psi\|_{\mathcal{L}} \leq C_T \|\varphi - \psi\|_{\mathcal{L}}, \quad \forall \varphi, \psi \in \mathcal{L} \quad (33)$$

The last inequality means the continuity of the evolution operator S_t in the norm of the space \mathcal{L} (see (29) and compare with (30)).

Remark 4. It should be noted that the evolution operator and, more generally, the time-shift is not a (strongly) continuous mapping in the norm of the space \mathcal{L} (see (9)). This can be illustrated by the following simple (scalar) example.

Consider the space

$$\mathcal{Lip}([-r, T]; \mathbb{R}) \equiv \left\{ v : [-r, T] \rightarrow \mathbb{R} : \sup \left\{ \frac{|v(s) - v(t)|}{|s - t|}, s \neq t; s, t \in [-r, T] \right\} < \infty \right\}$$

and analogously define the space $\mathcal{Lip}([-r, 0]; \mathbb{R})$ with the natural norm

$$\|v\|_{\mathcal{Lip}} \equiv \max_{\theta \in [-r, 0]} |v(\theta)| + \sup \left\{ \frac{|v(s) - v(t)|}{|s - t|}, s \neq t; s, t \in [-r, 0] \right\}.$$

The (strong) continuity of the time-shift means that

$$\forall v \in \mathcal{Lip}([-r, T]; \mathbb{R}) \text{ and } \forall t \in [0, T] \implies \lim_{h \rightarrow 0} \|v_{t+h} - v_t\|_{\mathcal{Lip}} = 0. \quad (34)$$

Obviously, when $t = 0$ one considers $h \rightarrow 0^+$, while for $t = T$, the case $h \rightarrow 0^-$ should be investigated.

To prove the claim, we must show that (34) does not hold, i.e.

$$\exists v \in \mathcal{Lip}([-r, T]; \mathbb{R}) \text{ and } \exists t_0 \in [0, T] \text{ for which } \lim_{h \rightarrow 0} \|v_{t_0+h} - v_{t_0}\|_{\mathcal{Lip}} \neq 0. \quad (35)$$

Thus, consider the case $t_0 = 0, h \rightarrow 0^+$ and the function

$$v(t) \equiv \begin{cases} 0, & t \in [-r, 0] \\ t, & t \in (0, T] \end{cases}.$$

It can be seen that $v_{t_0} = v_0 \equiv 0$ and

$$v_{t_0+h} = v_{t_0+h}(\theta) = \begin{cases} 0, & \theta \in [-r, -h] \\ h + \theta, & \theta \in (-h, 0] \end{cases}.$$

Hence, $\|v_{t_0+h} - v_{t_0}\|_{\mathcal{L}ip} = \|v_{t_0+h}\|_{\mathcal{L}ip} = h + 1$ and finally $\lim_{h \rightarrow 0^+} \|v_{t_0+h} - v_{t_0}\|_{\mathcal{L}ip} = \lim_{h \rightarrow 0^+} (h + 1) = 1 \neq 0$, which means that (34) does not hold. In the space \mathcal{L} , we would proceed analogously. \square

Remark 5. In the same way as in the previous remark one can show that the time-shift is **not** a (strongly) continuous mapping in the topology of $L^\infty(-r, 0)$. One could consider the

function $\tilde{v}(t) \equiv \begin{cases} 0, & t \in [-r, 0] \\ 1, & t \in (0, T] \end{cases}$ and $t_0 = 0$ to show that $\lim_{h \rightarrow 0^+} \|\tilde{v}_h - \tilde{v}_0\|_{L^\infty(-r, 0)} = 1 \neq 0$.

By the way, $\tilde{v} = \frac{d}{dt}v$, where, as usually, the time-derivative is understood in the sense of distributions. \square

The above remarks show that despite of the existence and uniqueness of solutions in the space \mathcal{L} and even strong continuity of the evolution operator S_t in the norm of \mathcal{L} (see (33)), the pair $(S_t; \mathcal{L})$ does *not* form a *dynamical system* since S_t is not strongly continuous as a mapping of time variable.

The methods, developed for ordinary delay equations in [33] suggest to restrict our considerations to a smaller subset of the space of Lipschitz functions. In this paper we follow this suggestion and consider the evolution operator S_t on the following subset of \mathcal{L}

$$X \equiv \{ \varphi \in C^1([-r, 0]; D(A^{-1/2})) \quad \text{such that}$$

$$\varphi(0) \in D(A^{1/2}) \text{ and } \dot{\varphi}(0) + A\varphi(0) + d\varphi(0) = F_1(\varphi) \} \subset \mathcal{L}. \quad (36)$$

Here the equality $\dot{\varphi}(0) + A\varphi(0) + d\varphi(0) = F_1(\varphi)$ is understood as an equality in $D(A^{-1/2})$.

Remark 6. The set X is an analogue of the solution manifold introduced in [33] for the case of ODEs with state-dependent delays. \square

To show that the set X is invariant under the evolution operator S_t , we first have to establish an additional smoothness property of the solutions of problem (1), (5).

Lemma 3. For any $\varphi \in C^1([-r, 0]; D(A^{-1/2}))$ such that $\varphi(0) \in D(A^{1/2})$, the solution to (1), (5) (which is given by Theorem 1) has the property (c.f. Proposition 1 and Theorem 1)

$$\dot{u} \in C([0, T]; D(A^{-1/2})), \quad \forall T > 0. \quad (37)$$

Remark 7. We do not assume $\varphi(0) \in D(A)$, just $\varphi(0) \in D(A^{1/2})$, so we cannot directly use [19, Theorem 3.5, p.114].

Proof of Lemma 3. By Proposition 1 and Theorem 1, for any $\varphi \in C^1([-r, 0]; D(A^{-1/2}))$ such that $\varphi(0) \in D(A^{1/2})$, there exists a unique solution $u(t) \in C([-r, T]; D(A^{-1/2})) \cap C([0, T]; D(A^{1/2}))$. This property and Lemma 1 then imply the continuity of the function

$$p(t) \equiv F_1(u_t) \in C([0, T]; L^2(\Omega)). \quad (38)$$

Consider the following auxiliary *linear system without delay*

$$\begin{cases} \dot{v}(t) + Av(t) + dv(t) = p(t), & t \geq 0, \\ v(0) = \varphi(0) \in D(A^{1/2}) \end{cases} \quad (39)$$

In the same way as in (17), the Galerkin approximate solution $v^m = v^m(t, x) = \sum_{k=1}^m g_{k,m}(t)e_k$ of order m to (39) can be defined such that

$$\begin{cases} \langle \dot{v}^m + Av^m + dv^m - p(t), e_k \rangle = 0, & t \geq 0, \\ \langle v^m(0), e_k \rangle = \langle \varphi(0), e_k \rangle, & \forall k = 1, \dots, m. \end{cases} \quad (40)$$

where $g_{k,m} \in C^1(0, T; \mathbb{R}) \cap L^2(-r, T; \mathbb{R})$ and $\dot{g}_{k,m}(t)$ is absolutely continuous.

The difference between approximate solutions u^m and v^m lies in that v^m are solutions just to *linear system* (40). So, for any two approximate solutions v^n and v^m (solutions to (40) of different orders n and m), one has $g_{k,n}(t) \equiv g_{k,m}(t)$, which is denoted by $g_k(t)$.

Multiply (40) by $\lambda_k g_k$ and sum for $k = n+1, \dots, n+p$ (p is any positive integer) to get

$$\begin{aligned} & \langle \dot{v}^{n+p}(t) - \dot{v}^n(t), A(v^{n+p}(t) - v^n(t)) \rangle + \|A(v^{n+p}(t) - v^n(t))\|^2 + \\ & + d \langle v^{n+p}(t) - v^n(t), A(v^{n+p}(t) - v^n(t)) \rangle = \langle (P_{n+p} - P_n)p(t), A(v^{n+p}(t) - v^n(t)) \rangle \end{aligned}$$

It should be recalled that, see the proof of Theorem 1, P_m is the orthogonal projection onto the subspace $\text{span}\{e_1, \dots, e_m\} \subset L^2(\Omega)$. Hence,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \|A(v^{n+p}(t) - v^n(t))\|^2 + d \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 \leq \\ \leq \| (P_{n+p} - P_n)p(t) \|^2 + \frac{1}{2} \| (P_{n+p} - P_n)p(t) \|^2 + \\ + \frac{1}{2} \|A(v^{n+p}(t) - v^n(t))\|^2 \end{aligned}$$

which gives

$$\frac{d}{dt} \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \|A(v^{n+p}(t) - v^n(t))\|^2 \leq \| (P_{n+p} - P_n)p(t) \|^2.$$

Integrating the last estimate results ($\forall t \in [0, T]$) in

$$\begin{aligned} \|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \int_0^t \|A(v^{n+p}(\tau) - v^n(\tau))\|^2 d\tau \leq \\ \leq \|A^{\frac{1}{2}}(v^{n+p}(0) - v^n(0))\|^2 + \int_0^t \| (P_{n+p} - P_n)p(\tau) \|^2 d\tau \leq \\ \leq \| (P_{n+p} - P_n)A^{\frac{1}{2}}\varphi(0) \|^2 + \int_0^t \| (P_{n+p} - P_n)p(\tau) \|^2 d\tau. \end{aligned}$$

Summing up, the above estimate, the fact that $\varphi(0) \in D(A^{1/2})$, the strong convergence $\|I - P_n\| \rightarrow 0$ for $n \rightarrow \infty$, and (38) imply that

$$\text{the sequence } \{v^n\}_{n=1}^\infty \text{ is a Cauchy sequence in } C([0, T]; D(A^{\frac{1}{2}})). \quad (41)$$

Now our goal is to show that the sequence $\{\dot{v}^n\}_{n=1}^\infty$ is a Cauchy sequence in $C([0, T]; D(A^{-1/2}))$. So, multiply first (40) by $\lambda_k^{-\frac{1}{2}}$ to get $\lambda_k^{-\frac{1}{2}}\dot{g}_k(t) = -\lambda_k^{\frac{1}{2}}g_k(t) - d\lambda_k^{-\frac{1}{2}}g_k(t) + \langle \lambda_k^{-\frac{1}{2}}p(t), e_k \rangle$. This gives $\lambda_k^{-1}(\dot{g}_k(t))^2 \leq 3\lambda_k(g_k(t))^2 + 3d^2\lambda_k^{-1}(g_k(t))^2 + 3|\langle \lambda_k^{-\frac{1}{2}}p(t), e_k \rangle|^2$. The sum for $k = n+1, \dots, n+p$ reads

$$\begin{aligned} \|A^{-\frac{1}{2}}(\dot{v}^{n+p}(t) - \dot{v}^n(t))\|^2 \leq 3\|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \\ + 3d^2\|A^{-\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \frac{3}{\lambda_{n+1}}\|(P_{n+p} - P_n)p(t)\|^2 \leq \\ \leq 3\left(1 + \frac{d^2}{\lambda_{n+1}^2}\right)\|A^{\frac{1}{2}}(v^{n+p}(t) - v^n(t))\|^2 + \frac{3}{\lambda_{n+1}}\|I - P_n\|^2\|p(t)\|^2. \end{aligned}$$

The last estimation together with (41) give that

$$\text{the sequence } \{\dot{v}^n\}_{n=1}^\infty \text{ is a Cauchy sequence in } C([0, T]; D(A^{-\frac{1}{2}})). \quad (42)$$

Thus, there exists a unique solution $v(t)$ ($v \equiv \lim_{n \rightarrow \infty} v^n$) to the linear system (39), which satisfies $v \in C([0, T]; D(A^{\frac{1}{2}}))$ and $\dot{v} \in C([0, T]; D(A^{-\frac{1}{2}}))$.

On the other hand, the nonlinear delay system (1), (5) with the initial function φ has also a unique solution. From the construction of $p(t)$ (see (38)), it follows that $u(t) \equiv v(t)$ for all $t \in [0, T]$, which gives (37) and completes the proof of Lemma 3. \square

Lemma 3 particularly shows that the set X , defined by (36), is invariant under the evolution operator S_t (see (29)). This fact allows to define an evolution operator (denoted again by S_t) $S_t : X \rightarrow X$ in the same way as in (29). Now, if the natural norm

$$\|\varphi\|_X \equiv \max_{s \in [-r, 0]} \|A^{-1/2}\varphi(s)\| + \max_{s \in [-r, 0]} \|A^{-1/2}\dot{\varphi}(s)\| + \|A^{1/2}\varphi(0)\|$$

on X is taken into account, then Theorem 1, Lemma 3, and Proposition 1 give the continuity of S_t with respect to t in the norm of X . Hence, $(S_t; X)$ defines a dynamical system.

Now we will pay attention to the long-time asymptotic behavior of the constructed evolution semigroup $S_t : X \rightarrow X$.

Theorem 2. Using the above notation and under the assumptions of Theorem 1, the dynamical system (S_t, X) is dissipative. If, in addition, $q = 0$ in **(H1.η)**, then (S_t, X) possesses a compact global attractor \mathcal{A} , which is a bounded set in the space $C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^\alpha))$, $\alpha \in (\frac{1}{2}, 1)$.

Proof of Theorem 2. It will be shown first that (S_t, X) is a dissipative dynamical system. To that end, the below proposition is needed.

Proposition 3. [25, Lemma 1] Let all the assumptions of Theorem 1 hold and let $\alpha \in (\frac{1}{2}, 1)$. Then there exists a bounded subset \mathcal{BV}_α of the space $C^1([-r, 0]; D(A^{-\frac{1}{2}})) \cap C([-r, 0]; D(A^\alpha))$, which absorbs any strong solution to the problem (1) and (5) for any initial function $\varphi \in \mathcal{L}$.

Second, to apply the classical theorem on the existence of a global attractor (see, for example [2, 30, 6]), we show that (S_t, X) is asymptotically compact. Consider therefore any solution $u(t)$ to the problem (1) and (5) with $\varphi \in \mathcal{BV}_\alpha$ as an initial function. We will show that for any $\delta > r > 0$ and any $T > \delta$ the set $\mathcal{U} \equiv \{u_t = S_t\varphi \mid \varphi \in \mathcal{BV}_\alpha, t \in [\delta, T]\}$ is relatively compact in X .

Recall that the set \mathcal{BV}_α is a ball in $C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^\alpha))$ (for more details see [25]) and notice that, by Corollary 4 from [27], the set \mathcal{BV}_α is relatively compact

in $C([-r, 0]; D(A^{-1/2}))$ (see also [27, lemma 1]). It remains to show that $\{\dot{u}(t) \mid \varphi \in \mathcal{BV}_\alpha, t \in [\delta - r, T]\}$ is equi-continuous in $C([\delta - r, T]; D(A^{-1/2}))$.

Proposition 4. [19, Corollary 4.3.3 and Theorem 4.3.5]. Let A be an infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$. If $f \in L^1((0, T); Y)$ is locally Hölder continuous on $(0, T]$, then for every $x \in Y$ the initial value problem

$$\dot{u}(t) = Au(t) + f(t), t > 0; \quad u(0) = x$$

has a unique solution u . If $f \in C^\theta([0, T]; Y)$, then for every $\delta > 0$, $Au \in C^\theta([\delta, T]; Y)$ and $\dot{u} \in C^\theta([\delta, T]; Y)$.

Here $C^\theta([0, T]; Y)$ denotes the family of all Hölder continuous functions on $[0, T]$ with the exponent $\theta \in (0, 1)$. In this case, $Y = L^2(\Omega)$.

In order to apply Proposition 4 to our case, we have to show that $p(t) = F_1(u_t) \in C^\theta([\delta - r, T]; L^2(\Omega))$ (c.f. (38),(39)). Therefore, consider $t \in [\delta - r, T]$ and

$$\begin{aligned} \|p(t+h) - p(t)\| &= \|F_1(u_{t+h}) - F_1(u_t)\| \leq \\ &\leq L_{F_1}[\ell_{\mathcal{BV}_\alpha}] \max_{s \in [-r, 0]} \|A^{-1/2}(u(t+h+s) - u(t+s))\| \leq L_{F_1}[\ell_{\mathcal{BV}_\alpha}] \ell_{\mathcal{BV}_\alpha} |h| \end{aligned}$$

where $L_{F_1}[\ell_{\mathcal{BV}_\alpha}]$ is the constant defined in Lemma 1 with $\ell_{\mathcal{BV}_\alpha}$ such that $\|\psi\| \leq \ell_{\mathcal{BV}_\alpha} \forall \psi \in \mathcal{BV}_\alpha$ (the existence of such $\ell_{\mathcal{BV}_\alpha}$ follows from Proposition 2). Here, $q = 0$ is used.

The last inequality shows that $p : [\delta - r, T] \rightarrow L^2(\Omega)$ is Lipschitz continuous, which is the situation to which Proposition 3 can be applied. It should also be noted that the family $\{p(t)\}$, for all initial $\varphi \in \mathcal{BV}_\alpha$, is uniformly Lipschitz, i.e. all the Lipschitz constants are lower or equal to $L \equiv L_{F_1}[\ell_{\mathcal{BV}_\alpha}] \cdot \ell_{\mathcal{BV}_\alpha}$. Then by Proposition 2, it is guaranteed (see the proof) that the family $\{\dot{u}(t) \mid \varphi \in \mathcal{BV}_\alpha, t \in [\delta - r, T]\}$ is uniformly Hölder continuous, and thus equi-continuous in $C([\delta - r, T]; L^2(\Omega))$.

Proposition 5. [27, lemma 1] Let B be a Banach space. A set F of $C([0, T]; B)$ is relatively compact if and only if

- (i) $F(t) \equiv \{f(t) : f \in F\}$ is relatively compact in B , $0 < t < T$,
- (ii) F is uniformly equicontinuous, i.e. $\forall \varepsilon > 0, \exists \eta$ such that $\|f(t_2) - f(t_1)\|_B \leq \varepsilon, \forall f \in F, \forall 0 \leq t_1 \leq t_2 \leq T$ such that $|t_2 - t_1| \leq \eta$

Applying Proposition 5 completes the proof of Theorem 2. \square

As an application we can consider the diffusive Nicholson blowflies equation (see e.g. [29]) with state-dependent delays, i.e. the equation (1) where $-A$ is the Laplace operator with the Dirichlet boundary conditions, $\Omega \subset \mathbb{R}^{n_0}$ is a bounded domain with a smooth boundary, the nonlinear (birth) function b is given by $b(w) = p \cdot w e^{-w}$. The function b is bounded, so for any delay function η satisfying $(H1.\eta)$, the conditions of Theorem 1 and Theorem 2 are satisfied. As a result, we conclude that the initial value problem (1) and (5) is well-posed in X and the dynamical system (S_t, X) has a global attractor (Theorem 2).

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